John Holden
Tutonka3000@aol.com

Inverting the Fibonacci Sequence

Mathematicians have been fascinated by the majestic simplicity of the *Fibonacci Sequence* for centuries. It starts as a simple $1,1,2,3,5,8,13,...$ computed *recursively,* each term is equivalent to the sum of the previous two terms. This can be expressed algebraically as $F_{n+2} = F_{n+1} + F_n$ provided $n \geq 1$.

Fibonacci is so simple that children in their first algebra classes are drawn to ponder the existence of a formula that more concisely defines the sequence. Graphing it indicates an exponential correlation, and indeed nineteenth century mathematician J. P. M. Binet discovered that the Golden Mean was related to the Fibonacci Sequence by proving that $F_n = \frac{\phi^n - \bar{\phi}^n}{\phi - \bar{\phi}}$, provided that ϕ is the Golden Mean and equal in value to $\frac{1+\sqrt{5}}{2}$. $\bar{\phi}$ is the conjugate, $\frac{1-\sqrt{5}}{2}$. This result is well known so we need not go further into Binet's formula here.

When looking at a graph of this sequence, I pondered the existence of an inverse function that could compute the value of n, the index number which defines each term's position among the sequence, from the original Fibonacci term, F_n.

Finding an inverse for Binet's formula is an algebraic nightmare, and it seems obvious that there cannot be a perfect inverse function because each ordinate does not have a unique abscissa-specifically $F_1 = F_2 = 1$. So the inverse function will have some restrictions to its domain because Binet's formula does not provide a one-to-one function.

By chance I stumbled across this theoretic inverse function. It reads
$Fib^{-1}(n) = n = \lfloor \log_\phi F_n \rfloor + 2 \quad (n = 2,3,4,...)$.

Try a few Fibonacci numbers yourself. Use Binet's formula to find the nth term, then use the new inverse function to find the index number, n, which should be the same as the first n. We already understand that the inverse function will not work for $n = 1$ because $F_1 = F_2 = 1$ (notice that this means the function does work for $n = 2$).

How can we justify this formula for all integers n greater than 1?

Proof

The proof is a simple induction argument that hinges on the recursive properties of both the Fibonacci Sequence itself and the Golden Mean.

What recursive property of the Golden Mean? Solve the equation $x^{n+2} = x^{n+1} + x^n$ for all integer n. The exact values of the Golden Mean and its conjugate are given by this formula, so $\phi^{n+2} = \phi^{n+1} + \phi^n$ for all integers n.

If we deconstruct the rounding in our inverse function, $n = \lfloor \log_\phi F_n \rfloor + 2$, we will find that $\phi^{n-2} \le F_n < \phi^{n-1}$. Actually, ϕ^{n-2} is equal to F_n only when $n = 2$.
Besides, what we need to do to justify the inverse function is prove the inequality $\phi^{n-2} \le F_n < \phi^{n-1}$ for $n = 2, 3, 4, \ldots$.

This justification is simple when using Mathematical Induction knowing the recursive properties of phi and the Fibonacci Sequence. To do this, we tear the inequality into two parts of $\phi^{n-2} < F_n$ and $F_n < \phi^{n-1}$ to justify separately (it is obvious that $\phi^{n-2} < \phi^{n-1}$ for all n and we do not need to discuss this).

1. When $n = 2$, we see that this inequality holds with $\phi^{n-2} = F_n$. We will quickly see that from this point on $\phi^{n-2} < F_n$.

$$1 = \phi^0 = F_2 < \phi^1 = 1.618\ldots$$

2. Now let us prove $\phi^{n-2} < F_n$ for all integer $n \ge 3$ by the Principle of Mathematical Induction.

   Basis Step:

   When $n = 3$, then

$$\phi^1 = 1.618\ldots < 2 = F_3,$$

   and when $n = 4$, then

$$\phi^2 = 2.618\ldots < 3 = F_4.$$

   Induction Step:

   Our induction for this theorem hinges on the recursive properties of the Golden Mean and Fibonacci sequence which state that $\phi^n = \phi^{n-1} + \phi^{n-2}$ and

$F_n = F_{n-1} + F_{n-2}$. With these properties and the demonstration of the inequality for two consecutive values of n we can establish $\phi^{n-2} < F_n$ for all integer $n \geq 3$. Let n be a positive integer greater than 4 and consider ϕ^{n-2}. The recursion formula for ϕ gives

$$\phi^{n-2} = \phi^{n-3} + \phi^{n-4}$$

Appling the induction hypothesis, $\phi^{k-2} < F_k$, is true for $n = 3,4$. For $k = 5$ it follows that

$$\begin{array}{c} \phi^{k-2} < F_{k-2} \\ \underline{+\phi^{k-1} < +F_{k-1}} \\ \phi^{k} < F_{k} \end{array}$$

Which is what we were trying to establish. This recursion can be repeated for all $n \geq 3$, thus this inequality is true on that interval. We have shown that for all integer $n \geq 3$, $\phi^{n-2} < F_n$ by the Principle of Mathematical Induction.

3. Now we must establish $F_n < \phi^{n-1}$ for all integer $n \geq 2$ through mathematical induction.

   Basis Step:

   When $n = 2$, then

$$F_2 = 1 < 1.618\ldots = \phi^1$$

   and when $n = 3$, then

$$F_3 = 2 < 2.618\ldots = \phi^2.$$

   Induction Step:

Let n be a positive integer greater than 3. Appling the Induction Hypothesis, $F_k < \phi^{k-1}$, and consider $k = n-1$ and $k = n-2$. For $k = 4$, $F_4 < \phi^{4-1}$ because

$$\begin{array}{r} F_{k-2} < \phi^{k-3} \\ + F_{k-1} < +\phi^{k-2} \\ \hline F_k < \phi^{k-1} \end{array}$$

Again, we can repeat this recursion to establish $F_k < \phi^{k-1}$ for all integer $k \ge 5$.

Therefore, by the Principle of Mathematical Induction, it follows that, for all $n \ge 3$,

$$F_n < \phi^{n-1}$$

holds.

4. These induction arguments prove the theorem.

Using this Formula

This formula is primarily an inverse function for Fibonacci numbers. Analogous to this operation are inverse trigonometric functions. Just as $\sin(\arcsin(x)) = x$, we could express Binet's Formula as $\mathrm{Fib}(n) = x$ and let the Inverse Formula be expressed in function notation as $\mathrm{Fib}^{-1}(x) = n$, so $\mathrm{Fib}(\mathrm{Fib}^{-1}(x)) = x$ and $\mathrm{Fib}^{-1}(\mathrm{Fib}(n)) = n$ provided $n = 2, 3, 4, \ldots$.

If some number x, not in the Fibonacci sequence, is entered into the inverse function an integer n corresponding to the greatest Fibonacci number less than that x will be given, which immediately brings us to another use for this theorem.

Characteristic Function for the Fibonacci sequence

Because any number can be inserted into the inverse function, the index, n, produced can be used to determine if a number is a Fibonacci number, or *characteristic* of the Fibonacci sequence.

If $\text{Fib}(\text{Fib}^{-1}(x)) = x$ then x is a Fibonacci number.

Counting Function for the Fibonacci sequence

Another possible use for this function is to enter a number x into the inverse function and determine whether x is characteristic of the Fibonacci sequence. If it is not, then the output is equal to the number of Fibonacci numbers less than x. If x is characteristic of the Fibonacci sequence then the number of Fibonacci numbers less than it would be the output n would be one greater than the number of Fibonacci numbers less than x because that n includes x. This is analogous to the π *Counting Function* in the Prime Number Theorem.

Conclusion

We now have an inverse formula, a characteristic function, and a counting function for the Fibonacci sequence. This is a new operation that can be used along with Binet's Formula any time one teaches or researches the Fibonacci sequence. This summer I used the inverse formula when researching Fibonacci Primes. I wonder if Fibonacci himself might have used this formula with his rabbits to see how many generations had gone by.

References

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